

A product space of $\{0, 1\}$ and an abstract polycrystal

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Abstract

A partition of a $\Lambda(\text{Card}\Lambda \succ \aleph_0)$ -product space of $\{0, 1\}$ defines an abstract polycrystal composed of abstract singlecrystals a decomposition space (space of equivalence classes) of each of which is self-similar.

1 Introduction

Let $(X, \tau) = (\{0, 1\}^\Lambda, \tau_0^\Lambda)$, $\text{Card}\Lambda \succ \aleph_0$ (aleph zero) be the Λ -product space of $(\{0, 1\}, \tau_0)$ where τ_0 is a discrete topology for $\{0, 1\}$. The topological space (X, τ) needs not to be metrizable, that is, the relation $\text{Card}\Lambda \succ \aleph$ (aleph) may hold. In the present report, assuming the lattice point of crystal to be a map $x : \Lambda \rightarrow \{0, 1\}$ (for example, $\Lambda = \mathbf{N}, \mathbf{R}, \dots$), we will mathematically confirm the existence of a partition $\{X_1, \dots, X_n\}$ of X , $X_i \in (\tau \cap \mathfrak{S})^1 - \{\phi\}$, a decomposition space (i.e, a space of equivalence classes) of each element X_i of which is self-similar. The partition of X can be regarded as a kind of polycrystal² in an abstract sense each abstract singlecrystal $X_i = (X_i, \tau_{X_i})$ ³ of which is characterized by its self-similar decomposition space. Namely, all points in X_i are classified into equivalence classes and then the equivalence classes coalesce to form a self-similar structure.

Some easily verified statements are summarized in the next section for the preliminaries.

2 Preliminaries

2-a) Any zero-dimensional (0-dim), perfect, T_0 (necessarily T_2)-space (X, τ) has its partition $\{X_1, \dots, X_n\}$, $X_i \in (\tau \cap \mathfrak{S}) - \{\phi\}$ for any n . Here $X_i \cap X_{i'} = \phi$ for $i \neq i'$ and $\bigcup_{i \in \bar{n}} X_i = X$ ⁴. Each subspace (X_i, τ_{X_i}) is a 0-dim, perfect, T_2 -space.

2-b) Let X be a 0-dim, perfect, compact T_2 -space. Then, for any compact metric space Y , there exists a continuous map f from X onto Y .

2-c) If $f : (X, \tau) \rightarrow (Y, \tau')$ is a quotient map, then the map $h : (Y, \tau') \rightarrow (\mathcal{D}_f, \tau(\mathcal{D}_f))$, $y \mapsto f^{-1}(y)$ is a homeomorphism. Here, the decomposition \mathcal{D}_f of X and the decomposition topology $\tau(\mathcal{D}_f)$ are given by $\mathcal{D}_f = \{f^{-1}(y) \subset X; y \in Y\}$ and $\tau(\mathcal{D}_f) = \{\mathcal{U} \subset \mathcal{D}_f; \bigcup \mathcal{U} \in \tau\}$, respectively.

¹The notaion $\tau \cap \mathfrak{S}$ denotes the set of all closed and open sets (clopen sets) of (X, τ) .

²We may replace a polycrystal with a tiling and a singlecrystal with a tile, respectively.

³ $\tau_A = \{u \cap A; u \in \tau\}$ for $A \subset X$.

⁴ $\bar{n} = \{1, \dots, n\}$.

2-d) A space which is homeomorphic to a self-similar space is self-similar.

2-e) Let (X, τ) be a 0-dim, perfect, compact, not-metrizable T_2 -space. From the Urysohn's metrization theorem (X, τ) is not second countable. In the partition $\{(X_1, \tau_{X_1}), \dots, (X_n, \tau_{X_n})\}$ of (X, τ) , there exists a number $i_0 \in \bar{n}$ such that $(X_{i_0}, \tau_{X_{i_0}})$ is not second countable. The subspace $(X_{i_0}, \tau_{X_{i_0}})$ is a 0-dim, perfect, compact not-metrizable T_2 -space.

3 A partition of the space $(\{0, 1\}^\Lambda, \tau_0^\Lambda)$

Since $(X, \tau) = (\{0, 1\}^\Lambda, \tau_0^\Lambda)$, $\tau_0 = 2^{\{0,1\}}$, $Card\Lambda \succ \aleph_0$ is easily verified to be a 0-dim, perfect, compact T_2 -space, from **2-a)** there exists a partition $\{(X_1, \tau_{X_1}), \dots, (X_n, \tau_{X_n})\}$ of (X, τ) where each (X_i, τ_{X_i}) is a 0-dim, perfect, compact T_2 -space.⁵ Then, from **2-b)**, there exists a continuous map f_i from (X_i, τ_{X_i}) onto any compact, self-similar metric space (Y, τ_d) . Since (X_i, τ_{X_i}) is a compact space and Y is a T_2 -space, the map $f : (X_i, \tau_{X_i}) \rightarrow (Y, \tau_d)$ is a quotient map. Therefore, from **2-c)**, the map $h : (Y, \tau_d) \rightarrow (\mathcal{D}_{f_i}, \tau(\mathcal{D}_{f_i})), y \mapsto f_i^{-1}(y)$ must be a homeomorphism. Since (Y, τ_d) is self-similar, according to **2-d)**, the decomposition space $(\mathcal{D}_{f_i}, \tau(\mathcal{D}_{f_i}))$ of (X_i, τ_{X_i}) is also self-similar. Here, we note that the decomposition space \mathcal{D}_{f_i} of X_i is not a trivial one $\{\{x\}; x \in X\}$ especially for a self-similar, connected space such as the Sierpiński carpet (S.B.Nadler Jr, 1992). In fact, disconnected space $\{\{x\}; x \in X\}$ (a decomposition space of X_i) is never homeomorphic to a connected space.

Regarding each subspace $(X_i, \tau_{X_i}), i \in \bar{n}$ an abstract singlecrystal we obtain an abstract polycrystal X composed of (X_i, τ_{X_i}) which has a self-similar decomposition space.

Finally, we note that in the above discussions we can replace the self-similar space with a compact substance in the materials science such as dendrite [1] and then, we can obtain an abstract polycrystal $(\{0, 1\}^\Lambda, \tau_0^\Lambda)$ composed of abstract singlecrystal whose decomposition space is characterized by a dendrite.

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References

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⁵If $Card\Lambda \succ \aleph$, as mentioned in **2-e)**, there exists 0-dim, perfect, compact, *not-metrizable* T_2 -space $(X_{i_0}, \tau_{X_{i_0}})$.